

A remark On Abelianized Absolute Galois Group of Imaginary Quadratic Fields

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Abstract

The main purpose of this paper is to extend results from [1] on isomorphism types of the abelianized absolute Galois group \mathcal{G}_K^{ab} , where K denotes imaginary quadratic field. In particular, we will show that if the class number h_K of an imaginary quadratic field K different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ is a fixed prime number p then there are only two isomorphism types of \mathcal{G}_K^{ab} which could occur. For instance, this result implies that imaginary quadratic fields of the discriminant D_K belonging to the set $\{-35, -51, -91, -115, -123, -187, -235, -267, -403, -427\}$ all have isomorphic abelian parts of their absolute Galois groups.

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1 Introduction

1.1 Motivation

Let K be a global field, i.e. either a finite extension of the field \mathbb{Q} of rational numbers or a field of functions on a smooth projective geometrically connected curve X over a finite field k . In the first case K is a number field and in the later case K is a global function field. An interesting question to ask is: what kind of information about K one could recover from various subgroups of the absolute Galois group $\mathcal{G}_K = \text{Gal}(K^{sep} : K)$ associated to K ? The famous theorem on Neukirch and Uchida states that the isomorphism types of \mathcal{G}_K considered as topological group determines the isomorphism class¹ of the field K . A question concerning the abelian part \mathcal{G}_K^{ab} of \mathcal{G}_K has attracted much attention since the work [4] where in particular it was shown that there exists an example of imaginary quadratic fields with different class-groups and with isomorphic \mathcal{G}_K^{ab} . A dramatic improvement was achieved in the paper [1], where authors produced a lot of new examples of non-isomorphic imaginary quadratic fields which share the same isomorphism type of \mathcal{G}_K^{ab} and also showed that there are infinitely many isomorphism types of pro-finite groups which could occur as \mathcal{G}_K^{ab} . Moreover, based on their computations they made a conjecture that there are infinitely many imaginary quadratic fields with $\mathcal{G}_K^{ab} \simeq \widehat{\mathbb{Z}}^2 \times \prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, where $\widehat{\mathbb{Z}}$ denotes the group of pro-finite integers.

Motivated by the above results authors of the present paper started working on the question about isomorphism type of \mathcal{G}_K^{ab} where K denotes a global function field. For a global function field K of characteristic p with the exact constant field \mathbb{F}_q , $q = p^n$ we defined the invariant d_K as a natural number such that $n = p^k d_K$ with $\gcd(d_K, p) = 1$. Let $\text{Cl}^0(K)$ denotes the degree zero part of the class-group of K . In other words $\text{Cl}^0(K)$ is the abelian group of \mathbb{F}_q -rational points of the Jacobian variety associated to the curve X . In the pre-print [2] we proved the following result:

Theorem 1. *Suppose K and K' are two global function fields, then $\mathcal{G}_K^{ab} \simeq \mathcal{G}_{K'}^{ab}$ as pro-finite groups if and only if the following three conditions hold:*

1. K and K' share the same characteristic p ;
2. Invariants d_K and $d_{K'}$ coincide: $d_K = d_{K'}$;
3. The non p -parts of class-groups of K and K' are isomorphic:

$$\text{Cl}_{non-p}^0(K) \simeq \text{Cl}_{non-p}^0(K').$$

¹up to Frobenius twist in the case of function fields.

In particular, two function fields with the same exact constant field \mathbb{F}_q have isomorphic \mathcal{G}_K^{ab} if and only if they have isomorphic $\text{Cl}_{non-p}^0(K)$.

An important remark is our proof actually provides a description of isomorphism type of \mathcal{G}_K^{ab} . Let \mathcal{T}_K denotes the topological closure of the torsion of \mathcal{G}_K^{ab} . We showed that the isomorphism type of \mathcal{T}_K depends only on the cardinality q of the constant field of K and actually gave an explicit description of the group \mathcal{T}_K in terms of q only. We also constructed an isomorphism of abelian groups:

$$\text{Cl}_{non-p}^0(K) \simeq (\mathcal{G}_K^{ab}/\mathcal{T}_K)[\text{tors}].$$

In the proof of the theorem 1 we showed that the isomorphism type of \mathcal{G}_K^{ab} is determined by isomorphism types of these two groups: \mathcal{T}_q and $\text{Cl}_{non-p}^0(K)$. The main step is to prove that given groups \mathcal{T}_K and $\text{Cl}_{non-p}^0(K)$ there exists a unique isomorphism type of a pro-finite abelian group \mathcal{D}_K such that the following holds:

1. There exists an exact sequence: $0 \rightarrow \mathcal{T}_K \rightarrow \mathcal{D}_K \rightarrow \text{Cl}_{non-p}^0(K) \rightarrow 0$;
2. All torsion elements of \mathcal{D}_K are in \mathcal{T}_K .

Finally, combining all results of our paper one has:

Corollary 1. *For a global function field K of characteristic p there exists an isomorphism of topological groups:*

$$\mathcal{G}_K^{ab} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\infty \times \mathcal{D}_K.$$

Our result in the imaginary quadratic field case is quite similar to this statement.

1.2 The Statement of the Theorem

The main purpose of the present paper is to use technique from [2] in order to extend results of the paper [1]. Let K be an imaginary quadratic field different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$. Let $\mathcal{T} = \prod_n \mathbb{Z}/n\mathbb{Z}$ and let $\text{Cl}(K)$ denotes the ideal class group of K . Summarising results of [1] we have:

Theorem 2. *In the above settings the following holds:*

1. There exists an exact sequence of topological groups: $0 \rightarrow \widehat{\mathbb{Z}}^2 \times \mathcal{T} \rightarrow \mathcal{G}_K^{ab} \rightarrow \text{Cl}(K) \rightarrow 0$;
2. The topological closure $\overline{\mathcal{G}_K^{ab}[\text{tors}]}$ of the torsion subgroup of \mathcal{G}_K^{ab} is \mathcal{T} ;
3. The torsion subgroup of the quotient $\mathcal{G}_K^{ab}/\mathcal{T}$ is trivial if and only if $\mathcal{G}_K^{ab} \simeq \widehat{\mathbb{Z}}^2 \times \mathcal{T}$;

4. *There exist an injective map from $(\mathcal{G}_K^{ab}/\mathcal{T})[\text{tors}]$ to $\text{Cl}(K)$ and an algorithm with input K and output whether the group $(\mathcal{G}_K^{ab}/\mathcal{T})[\text{tors}]$ is trivial or not.*

Proof. See theorem 3.5, 4.4 and 5.1 from [1]. \square

Let us call the image of $(\mathcal{G}_K^{ab}/\mathcal{T})[\text{tors}]$ in $\text{Cl}(K)$ as $\text{Cl}^{split}(K)$. Roughly speaking our main result states that isomorphism type of \mathcal{G}_K^{ab} is uniquely determined by the isomorphism type of $\text{Cl}^{split}(K)$. More concretely, we will prove that given groups \mathcal{T} and $\text{Cl}^{split}(K)$ there exists a unique isomorphism type of a pro-finite abelian group \mathcal{D}_K such that the following holds:

1. There exists an exact sequence: $0 \rightarrow \mathcal{T} \rightarrow \mathcal{D}_K \rightarrow \text{Cl}^{split}(K) \rightarrow 0$;
2. All torsion elements of \mathcal{D}_K are in \mathcal{T} .

Then the main result of the present paper could be stated as:

Theorem 3. *Let K be an imaginary quadratic field different from $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$. There exists an isomorphism of topological groups $\mathcal{G}_K^{ab} \simeq \mathcal{D}_K \times \widehat{\mathbb{Z}}^2$.*

The above theorem extends results of the theorem 2 as follows:

Corollary 2. *For a fixed prime number p and an imaginary quadratic field K with class-number $h_K = p$ there are only two isomorphism types of \mathcal{G}_K^{ab} which could occur: either $\text{Cl}^{split}(K) = 0$ or $\text{Cl}^{split}(K) \simeq \mathbb{Z}/p\mathbb{Z}$. In particular, it was shown in [1] that imaginary quadratic fields with the discriminant D_K occurring in the list $\{-35, -51, -91, -115, -123, -187, -235, -267, -403, -427\}$ all have class-number 2 and have non-trivial $\text{Cl}^{split}(K)$, therefore they all share the same isomorphism class of \mathcal{G}_K^{ab} .*

2 The Proof

Our goal in this section is to prove theorem 3. We will do this in two steps. First, we will show that there exist a pro-finite group \mathcal{D}_K and an isomorphism $\mathcal{G}_K^{ab} \simeq \mathcal{D}_K \times \widehat{\mathbb{Z}}^2$. Then we will show that the group \mathcal{D}_K is uniquely determined by the isomorphism class of the abelian group $\text{Cl}^{split}(K)$, provided $K \neq \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$.

Since each pro-finite abelian group is isomorphic to the limit of finite abelian groups, by the Chinese remainder theorem we have that it is also isomorphic to the product over prime numbers of its primary components. We will work with these components separately instead of working with the whole group.

2.1 Proof of Splitting

Consider the exact sequence mentioned in the theorem 2:

$$0 \rightarrow \widehat{\mathbb{Z}}^2 \times \mathcal{T} \rightarrow \mathcal{G}_K^{ab} \rightarrow \text{Cl}(K) \rightarrow 0 \quad (1)$$

Taking a prime number l we get the following exact sequence of pro- l abelian groups:

$$0 \rightarrow \mathbb{Z}_l^2 \times \mathcal{T}_l \rightarrow \mathcal{G}_{K,l}^{ab} \rightarrow \text{Cl}_l(K) \rightarrow 0, \quad (2)$$

where $\mathcal{T}_l = \prod_{k \in \mathbb{N}} \mathbb{Z}/l^k \mathbb{Z}$ and \mathbb{Z}_l denotes the group of l -adic integers. If $\text{Cl}_l(K)$ is the trivial group then obviously $\mathcal{G}_{K,l}^{ab} \simeq \mathbb{Z}_l^2 \times \mathcal{T}_l$. Our goal is to describe the isomorphism type of $\mathcal{G}_{K,l}^{ab}$ in the case where $\text{Cl}_l(K)$ is not the trivial group. By the theorem 2 we know that \mathcal{T}_l is the closure of the torsion subgroup of $\mathcal{G}_{K,l}^{ab}$. Note that \mathcal{T}_l is a closed subgroup and hence the quotient is also pro- l group. Taking the quotient of the sequence 2 by \mathcal{T}_l we obtain:

$$0 \rightarrow \mathbb{Z}_l^2 \rightarrow \mathcal{G}_{K,l}^{ab}/\mathcal{T}_l \rightarrow \text{Cl}_l(K) \rightarrow 0.$$

Since \mathbb{Z}_l is torsion free we have $(\mathcal{G}_{K,l}^{ab}/\mathcal{T}_l)[\text{tors}]$ maps injectively to $\text{Cl}_l(K)$ which is finite. Denoting the group $\mathcal{G}_{K,l}^{ab}/\mathcal{T}_l$ by B_l we get isomorphism of topological groups²: $B_l \simeq B_l[\text{tors}] \oplus B'_l$, where B'_l denotes the non-torsion part of B_l . Since \mathbb{Z}_l is torsion free we also have the following exact sequence:

$$0 \rightarrow \mathbb{Z}_l^2 \rightarrow B'_l \rightarrow \text{Cl}_l(K)/\phi(B_l[\text{tors}]) \rightarrow 0.$$

Since B'_l is torsion free this exact sequence implies B'_l is a free \mathbb{Z}_l -module of rank two and hence $B'_l \simeq \mathbb{Z}_l^2$.

Let us denote the quotient map $\mathcal{G}_{K,l}^{ab} \rightarrow \text{Cl}_l(K)$ by ϕ . In notations from the introduction $\phi(B_l[\text{tors}]) = \text{Cl}_l^{\text{split}}(K)$. Consider the pre-image $\mathcal{D}_l \subset \mathcal{G}_{K,l}^{ab}$ of the group $\phi(B_l[\text{tors}]) \subset \text{Cl}_l(K)$. Note that \mathcal{D}_l is closed subgroup and we have the following exact sequence:

$$0 \rightarrow \mathcal{T}_l \rightarrow \mathcal{D}_l \rightarrow \phi(B_l[\text{tors}]) \rightarrow 0.$$

²This is true because $B_l[\text{tors}]$ is finite.

Summing up we have the following commutative diagram of pro- l abelian groups:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathbb{Z}_l^2 & \longrightarrow & B'_l & \longrightarrow & \text{Cl}_l(K)/\phi(B_l[\text{tors}]) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{T}_l \times \mathbb{Z}_l^2 & \longrightarrow & \mathcal{G}_{K,l}^{ab} & \xrightarrow{\phi} & \text{Cl}_l(K) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & \phi(B_l[\text{tors}]) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

Now consider the exact sequence coming from the medium column of the above diagram:

$$0 \rightarrow \mathcal{D}_l \rightarrow \mathcal{G}_{K,l}^{ab} \rightarrow B'_l \rightarrow 0.$$

We know that $B'_l \simeq \mathbb{Z}_l^2$, but \mathbb{Z}_l is a projective module and hence we could split this sequence to obtain isomorphism: $\mathcal{G}_{K,l}^{ab} \simeq \mathcal{D}_l \times B'_l \simeq \mathcal{D}_l \times \mathbb{Z}_l^2$, which finishes the first step.

2.2 Proof of Uniqueness

Our main result is to show that the group \mathcal{D}_l is determined uniquely by the isomorphism type of $\phi(B_l[\text{tors}]) = \text{Cl}^{split}_l(K)$. Consider the exact sequence:

$$0 \rightarrow \mathcal{T}_l \rightarrow \mathcal{D}_l \rightarrow \text{Cl}^{split}_l(K) \rightarrow 0$$

We know that the closure of the torsion subgroup of $\mathcal{G}_{K,l}^{ab}$ is \mathcal{T}_l therefore, \mathcal{D}_l contains no torsion elements apart from elements of \mathcal{T}_l . Our goal is to prove:

Theorem 4. *Given pro- l abelian group $\mathcal{T}_l \simeq \prod_{k \in \mathbb{N}} \mathbb{Z}/l^k \mathbb{Z}$ and a finite abelian l -group A there exists a unique isomorphism type of pro- l abelian group \mathcal{D}_l such that the following holds:*

1. *There exists an exact sequence of pro- l abelian groups: $0 \rightarrow \mathcal{T}_l \rightarrow \mathcal{D}_l \rightarrow A \rightarrow 0$;*
2. *The topological closure of the torsion subgroup of \mathcal{D}_l is \mathcal{T}_l : $\overline{(\mathcal{D}_l)[\text{tors}]} = \mathcal{T}_l$.*

The key idea in the proof is to use the Pontryagin duality for locally compact abelian groups to reduce the question about pro- l groups to the more elementary question about discrete torsion groups and then use the following theorem:

Theorem 5. *Let $\{C_i\}$ be a countable set of finite cyclic abelian l -groups with orders of C_i are not bounded as i tends to infinity and let A be any finite abelian l -group. Then up to isomorphism there exists a unique torsion abelian l -group B satisfying two following conditions:*

1. *There exists an exact sequence: $0 \rightarrow A \rightarrow B \rightarrow \bigoplus_{i \geq 1} C_i \rightarrow 0$;*
2. *A is the union of all divisible elements of B : $A = \bigcap_{n \geq 1} nB$.*

Proof. See [2]. □

We will show that the Pontryagin dual of the exact sequence $0 \rightarrow \mathcal{T}_l \rightarrow \mathcal{D}_l \rightarrow \text{Cl}^{split}(K) \rightarrow 0$ satisfies conditions of the theorem 5 and therefore \mathcal{D}_l is uniquely determined, since its Pontryagin dual $(\mathcal{D}_l)^\vee$ is uniquely determined.

2.2.1 The Pontryagin duality

We need to recall some properties of the Pontryagin duality for locally compact abelian groups. A good reference including some historical discussion is [3]. Let \mathbb{T} be the topological group \mathbb{R}/\mathbb{Z} given with the quotient topology. If A is any locally compact abelian group one consider the Pontryagin dual A^\vee of A which is the group of all continuous homomorphism from A to \mathbb{T} :

$$A^\vee = \text{Hom}(A, \mathbb{T}).$$

This group has the so-called compact-open topology and is a topological group. Here we list some properties of the Pontryagin duality we use during the proof:

1. The Pontryagin duality is a contra-variant functor from the category of locally compact abelian groups to itself;
2. If A is a finite abelian group treated with the discrete topology then $A^\vee \simeq A$ non-canonically;
3. We have the canonical isomorphism: $(A^\vee)^\vee \simeq A$;
4. The Pontryagin dual to the pro-finite abelian group A is a discrete torsion group and vice versa;
5. The Pontryagin duality sends direct products to direct sums and vice versa;

Having stated these facts we are able to finish the proof.

2.2.2 The final step

In a settings of the theorem 4 the multiplication by l^n map induces the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{T}_l[l^n] & \hookrightarrow & \mathcal{D}_l[l^n] & \xrightarrow{0} & A[l^n] \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow l^n & & \downarrow l^n & & \downarrow l^n \\
0 & \longrightarrow & \mathcal{T}_l & \longrightarrow & \mathcal{D}_l & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathcal{T}_l/l^n \mathcal{T}_l & \longrightarrow & \mathcal{D}_l/l^n \mathcal{D}_l & \longrightarrow & A/l^n A \longrightarrow 0
\end{array}$$

Since any torsion element x of \mathcal{D}_l is in \mathcal{T}_l the map from $\mathcal{D}_l[l^n]$ to $A[l^n]$ is the zero map and the map from $\mathcal{T}_l[l^n]$ to $\mathcal{D}_l[l^n]$ is an isomorphism. Now applying the Pontryagin duality to the above diagram we get:

$$\begin{array}{ccccccc}
0 & \longleftarrow & (\mathcal{T}_l[l^n])^\vee & \longleftarrow & (\mathcal{D}_l[l^n])^\vee & \xleftarrow{0} & (A[l^n])^\vee \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longleftarrow & (\mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l)^\vee & \longleftarrow & (A)^\vee \longleftarrow 0 \\
& & \uparrow l^n & & \uparrow l^n & & \uparrow l^n \\
0 & \longleftarrow & (\mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l)^\vee & \longleftarrow & (A)^\vee \longleftarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & (\mathcal{T}_l/l^n \mathcal{T}_l)^\vee & \longleftarrow & (\mathcal{D}_l/l^n \mathcal{D}_l)^\vee & \longleftarrow & (A/l^n A)^\vee \longleftarrow 0
\end{array}$$

Note that $(\mathcal{T}_l)^\vee$ is isomorphic to the direct sum of cyclic groups $(\mathcal{T}_l)^\vee \simeq \bigoplus_{k \in \mathbb{N}} \mathbb{Z}/l^k \mathbb{Z}$ and therefore $\bigcap_n l^n (\mathcal{T}_l)^\vee = \{0\}$. It means we have $(\bigcap_n l^n (\mathcal{D}_l)^\vee) \subset (A)^\vee$. Our goal is to show that $(\bigcap_n l^n (\mathcal{D}_l)^\vee) = (A)^\vee$.

Lemma 1. *Given any non-zero element x of $(A)^\vee \subset (\mathcal{D}_l)^\vee$ and any natural number n there exists an element $c_x \in (\mathcal{D}_l)^\vee$ such that $l^n c_x = x$.*

Proof. For fixed n consider the above diagram. Since the second row is exact the image of x in $(\mathcal{T}_l)^\vee$ is zero. Then its image in $(\mathcal{T}_l[l^n])^\vee$ is also zero. Since $(\mathcal{T}_l[l^n])^\vee \simeq (\mathcal{D}_l[l^n])^\vee$ it means that image of the non-zero element x in $(\mathcal{D}_l[l^n])^\vee$ is zero. Since the second column

is exact this means that x lies in the image of the multiplication by l^n map from $(\mathcal{D}_l)^\vee$ to $(\mathcal{D}_l)^\vee$ and therefore there exists c_x such that $l^n c_x = x$. \square

It means that we have proved:

Corollary 3. *The exact sequence $0 \leftarrow (\mathcal{T}_l)^\vee \leftarrow (\mathcal{D}_l)^\vee \leftarrow (A)^\vee \leftarrow 0$ satisfies conditions of the theorem 5.*

and therefore \mathcal{D}_l is uniquely determined since its Pontryagin dual $(\mathcal{D}_l)^\vee$ is uniquely determined by the theorem 5.

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